Memorandum From Amir Said, Aug. 4, 1995

1 Coefficients for Magnitude Quantization

With a given threshold T, we want to find the bias coefficients $\beta_N(T)$ for magnitudes in the interval $[NT, 2NT]$, where $N = 2^k$ is the number of subdivisions. Assuming that the magnitudes have a distribution with density $p(x)$, then

$$\begin{aligned}\beta_N(T) &= \frac{1}{\int_{NT}^{2NT} p(x)\,dx} \sum_{n=N}^{2N-1} \left[\int_{nT}^{(n+1)T} p(x)\,dx \left(\frac{\frac{1}{T}\int_{nT}^{(n+1)T} x\,p(x)\,dx}{\int_{nT}^{(n+1)T} p(x)\,dx} - n \right) \right] \\ &= \frac{\sum_{n=N}^{2N-1} \int_{nT}^{(n+1)T} (x - nT)\,p(x)\,dx}{T \int_{NT}^{2NT} p(x)\,dx} = \frac{\sum_{n=N}^{2N-1} \int_{0}^{T} x\,p(x+nT)\,dx}{T \int_{NT}^{2NT} p(x)\,dx}.\end{aligned}$$

Taking the average of $\beta_N(T)$, considering the probability that the coefficient belongs to the interval $[NT, 2NT]$, we obtain

$$\overline{\beta}_N = \frac{\int_0^\infty \frac{1}{T} \sum_{n=N}^{2N-1} \int_0^T x\,p(x+nT)\,dx\,dT}{\int_0^\infty \int_{NT}^{2NT} p(x)\,dx\,dT}. \tag{1}$$

Particularly, if we assume magnitudes with distribution

$$p(x) = \begin{cases} a\,e^{-ax}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{2}$$

then we have

$$\int_0^\infty \int_{NT}^{2NT} p(x)\,dx\,dT = \frac{1}{2aN}, \tag{3}$$

and

$$\frac{1}{T} \int_0^T x\,p(x+nT)\,dx = e^{-naT} \left(\frac{1 - e^{-aT}}{aT} - e^{-aT} \right). \tag{4}$$

By changing the order of summation and integration in (1), and with the substitution of the results above, it follows that

$$\begin{aligned}\overline{\beta}_N &= 2aN \sum_{n=N}^{2N-1} \left[\int_0^\infty \frac{e^{-naT} - e^{-(n+1)aT}}{aT}\,dT - \int_0^\infty e^{-(n+1)aT}\,dT \right] \\ &= 2N \left[\sum_{n=N}^{2N-1} \ln\left(\frac{n+1}{n}\right) - \sum_{n=N}^{2N-1} \frac{1}{i+1} \right].\end{aligned}$$

Thus, we can conclude that

$$\overline{\beta}_N = 2N\left[\ln 2 - \sum_{n=N+1}^{2N} n^{-1}\right] \tag{5}$$

$$= 2N\left[\ln 2 + \psi(N+1) - \psi(2N+1)\right], \tag{6}$$

where

$$\psi(n) = \frac{d\ln\Gamma(n)}{dn} \tag{7}$$

is the di-gamma function.

For $N = 2^k > 1$ this can be approximated by

$$\overline{\beta}_N \approx \tilde{\beta}_N = \frac{1}{2} - \frac{0.25}{(k+1)^2} \tag{8}$$

Below is a table with some values of $\overline{\beta}_N$ and $\tilde{\beta}_N$.

k	0	1	2	3	4	5
$N = 2^k$	1	2	4	8	16	32
$\overline{\beta}_N$	0.3863	0.4393	0.4690	0.4844	0.4922	0.4961
$\tilde{\beta}_N$	—	0.4375	0.4722	0.4844	0.4900	0.4931